$$\frac{a^8}{(b+c)(a-b)^2(a-c)^2}+\frac{b^8}{(a+c)(b-a)^2(b-c)^2} + \frac{c^8}{(a+b)(c-a)^2(c-b)^2} > 144\sqrt{3}r^3.$$

1064. *Proposed by Mircea Merca, University of Craiova, Romania.*

Let n be a positive integer. Prove that

$$0 < \frac{1}{\pi}\cdot\frac{2^n}{\binom{2n}{n}} - \sum_{k=1}^{n}\cos^{2n+1}\left(\frac{k\pi}{2n+1}\right) < 1.$$

1065. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $n \geq 0$ be an integer and let $\alpha, a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$ be complex numbers. Prove that

(a) $$\operatorname{Re}\left(\overline{\alpha}\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{2}\left(\sum_{k=0}^{n}|a_k|^2 + |\alpha|^2\sum_{k=0}^{n}|b_k|^2\right).$$

(b) $$\operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{2(n+1)}\left(\sum_{k=0}^{n}|a_k|^2 + \frac{(2n+1)(2n+3)}{3}\sum_{k=0}^{n}|b_k|^2\right).$$

SOLUTIONS

An algebraic inequality

1036. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a^3(a+1)+b^3(b+1)+c^3(c+1))\cdot\frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)} \geq 48.$$

Solution by Arkady Alt, San Jose, California.

We show that $(a^3(a+1)+b^3(b+1)+c^3(c+1)) \geq 6$ and

$$\frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)} \geq 8.$$

By the AM-GM inequality, we have

$$(a^3(a+1)+b^3(b+1)+c^3(c+1)) \geq 3\cdot\sqrt[4]{(abc)^4} + 3\cdot\sqrt[3]{(abc)^3} = 6.$$

On the other hand,

$$\frac{a^3+3}{a+1} \geq \frac{a+3}{2}.$$

Indeed, this is equivalent to $2a^3 - a^2 - 4a + 3 \geq 0$, or $(a-1)^2(2a+3) \geq 0$, which is clearly true. Furthermore, we have, again by AM-GM,

$$\prod_{\text{cyclic}} \frac{a+3}{2} = \frac{1}{8} \prod_{\text{cyclic}} (a+1+1+1) \geq 8 \prod_{\text{cyclic}} \sqrt[4]{a} = 8.$$

Also solved by ADNAN ALI, Atomic Energy Central School-4, India; MICHEL BATAILLE, Rouen, France; BRIAN BRADIE, Christopher Newport U.; JOHN CHRISTOPHER, California State U.–Sacramento; HABIB FAR, Lone Star C.; DMITRY FLEISCHMAN, Santa Monica, CA; SOOBIN HONG, Inst. of Sci. Education for the Gifted and Talented, Yonsei U., Korea; SI YOUNG KIM, Inst. of Sci. Education for the Gifted and Talented, Yonsei U., Korea; LAFAYETTE MATH CLUB, U. of Louisiana at Lafayette; ELIAS LAMPAKIS, Kiparissia, Greece; KEE-WAI LAU, Hong Kong, China; ABHAY MALIK, student, The Episcopal Acad.; PAOLO PERFETTI, Dipt. Mata., U. Roma Tor Vergata, Italy (2 solutions); NEW YORK MATH CIRCLE, City U. of New York; DIGBY SMITH, Mount Royal U.; RONALD SMITH, Florida Southwestern State C.; STAN WAGON, Macalester C.; and the proposer.

A trigonometric inequality

1037. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let P be a point inside the triangle ABC and let D, E, F be the projections of P on the sides BC, CA, and AB, respectively. Prove that

$$\frac{PA + PB + PC}{(EF \cdot FD \cdot DE)^{1/3}} \geq 2\sqrt{3}.$$

Solution by Michel Bataille, Rouen, France.

Since $\angle PFA = \angle PEA = 90^\circ$, PA is a diameter of the circumcircle of $\triangle PEF$. Hence $\frac{EF}{\sin A} = PA$ by the extended law of sines. Similarly, we have $\frac{FD}{\sin B} = PB$ and $\frac{DE}{\sin C}$ and so

$$\frac{PA + PB + PC}{(EF \cdot FD \cdot DE)^{1/3}} = \frac{PA + PB + PC}{(PA \cdot PB \cdot PC)^{1/3}(\sin A \sin B \sin C)^{1/3}}.$$

Now the AM-GM inequality gives

$$(PA \cdot PB \cdot PC)^{1/3} \leq \frac{PA + PB + PC}{3}$$

and

$$(\sin A \sin B \sin C)^{1/3} \leq \frac{\sin A + \sin B + \sin C}{3}$$

so it follows that

$$\frac{PA + PB + PC}{(EF \cdot FD \cdot DE)^{1/3}} \geq \frac{9}{\sin A + \sin B + \sin C}.$$

Lastly, since the sine function is concave on $(0, \pi)$, Jensen's inequality yields

$$\sin A + \sin B + \sin C \leq 3 \sin\left(\frac{A+B+C}{3}\right) = 3 \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$$